

GRADED CODIMENSIONS OF LIE SUPERALGEBRA  $b(2)$ 

DUŠAN REPOVŠ AND MIKHAIL ZAICEV

ABSTRACT. We study asymptotic behaviour of graded codimensions of Lie superalgebra  $b(2)$ . We prove that graded PI-exponent exists and is equal to  $3 + 2\sqrt{3}$ .

## 1. INTRODUCTION

We consider finite dimensional Lie superalgebras over a field of characteristic zero and study their  $\mathbb{Z}_2$ -graded identities. We pay main attention to numerical invariants of identities, in particular, to graded codimensions and their asymptotic behaviour.

It is well-known that in case  $\dim L < \infty$  both graded and ordinary codimensions are exponentially bounded ([2]). One of the more important questions of the theory of numerical invariants of polynomial identities is: does a (graded) PI-exponent exist?

There are many papers where the existence of PI-exponent is proved for different classes of algebras. For example, if  $A$  is an associative PI-algebra or a finite dimensional Lie, Jordan or alternative algebra then its PI-exponent exists and is a non-negative integer (see [4], [5], [6], [11]). The existence of PI-exponent for any finite dimensional simple algebra was proved in [8]. It is not difficult to show that if PI-exponent of  $A$  exists then it is less than or equal to  $d$  provided that  $d = \dim A < \infty$  (see for example [2]). In many important classes of algebras over an algebraically closed field (associative, Lie, Jordan, alternative) the equality  $\exp(A) = \dim A$  is equivalent to simplicity of  $A$  ([4], [5], [11]). Recently [8] it was shown that  $\exp(L) < \dim L$  for any finite dimensional simple Lie superalgebra  $L$  of the type  $b(t)$ ,  $t \geq 3$ , (in the notation of [9]). The existence of PI-exponent and similar inequality  $\exp(L) < \dim L$  for  $b(2)$  was also proved in [8] although it is not simple superalgebra.

Graded codimensions of Lie superalgebras were studied much less. In particular, it is still unknown if  $\exp^{gr}(L)$  exists even when  $L$  is a finite dimensional simple Lie superalgebra. In recent paper [10] it was proved that an upper graded PI-exponent of Lie superalgebra  $b(t)$ ,  $t \geq 2$ , is less than or equal to  $t^2 - 1 + t\sqrt{t^2 - 1}$ . In particular, this gives an upper bound for ordinary PI-exponent of  $b(t)$ . In the present paper we prove the existence of graded PI-exponent of  $b(2)$ . We also prove

---

2010 *Mathematics Subject Classification.* Primary 17B01, 16P90; Secondary 16R10.

*Key words and phrases.* Graded polynomial identities, Lie superalgebras, codimensions, exponential growth, PI-exponent.

The first author was supported by the Slovenian Research Agency grants P1-0292-0101, J1-6721-0101 and J1-5435-0101. The second author was partially supported by RFBR grant No 13-01-00234a. We thank the referee for comments.

that  $\exp^{gr}(b(2)) = 3 + 2\sqrt{3}$ . Note that it was recently announced that an ordinary PI-exponent does not exist in general non-associative case (see [12]).

## 2. MAIN CONSTRUCTIONS AND DEFINITIONS

Let  $L$  be a Lie superalgebra over a field  $F$  of characteristic zero, that is  $L = L_0 \oplus L_1$  is a non-associative  $\mathbb{Z}_2$ -graded algebra satisfying two identical relations

$$xy + (-1)^{|x||y|}yx = 0,$$

$$x(yz) = (xy)z + (-1)^{|x||y|}y(xz) = 0$$

where  $x, y, z \in L_0 \cup L_1$  and  $|x| = 0$  if  $x \in L_0$  while  $|x| = 1$  if  $x \in L_1$ .

Elements from  $L_0 \cup L_1$  are called *homogeneous* and we say that  $x$  is *even* if  $x \in L_0$  or  $x$  is *odd* if  $x \in L_1$ .

Denote by  $\mathcal{L}(X, Y)$  a free Lie superalgebra with infinite sets of even generators  $X$  and odd generators  $Y$ . A polynomial  $f = f(x_1, \dots, x_m, y_1, \dots, y_n) \in \mathcal{L}(X, Y)$  is said to be a *graded identity* of Lie superalgebra  $L = L_0 \oplus L_1$  if  $f(a_1, \dots, a_m, b_1, \dots, b_n) = 0$  whenever  $a_1, \dots, a_m \in L_0, b_1, \dots, b_n \in L_1$ .

Given positive integers  $0 \leq k \leq n$ , denote by  $P_{k, n-k}$  the subspace of all multilinear polynomials  $f = f(x_1, \dots, x_k, y_1, \dots, y_{n-k}) \in \mathcal{L}(X, Y)$  of degree  $k$  on even variables and of degree  $n - k$  on odd variables. Denote by  $Id^{gr}(L)$  the ideal of  $\mathcal{L}(X, Y)$  of all graded identities of  $L$ . Then  $P_{k, n-k} \cap Id^{gr}(L)$  is the subspace of all multilinear graded identities of  $L$  of total degree  $n$  depending on  $k$  even variables and  $n - k$  odd variables. Also denote by  $P_{k, n-k}(L)$  the quotient

$$P_{k, n-k}(L) = \frac{P_{k, n-k}}{P_{k, n-k} \cap Id^{gr}(L)}.$$

Then the graded  $(k, n - k)$ -codimension of  $L$  is

$$c_{k, n-k}(L) = \dim P_{k, n-k}(L)$$

and the total graded codimension of  $L$  is

$$c_n^{gr}(L) = \sum_{k=0}^n \binom{n}{k} c_{k, n-k}(L).$$

It is known (see [2]) that if  $\dim L < \infty$  then the sequence  $\{c_n^{gr}(L)\}_{n \geq 1}$  is exponentially bounded and one can consider the related sequence  $\sqrt[n]{c_n^{gr}(L)}$ . The latter sequence has the lower and upper limits

$$\underline{\exp}^{gr}(L) = \liminf_{n \rightarrow \infty} \sqrt[n]{c_n^{gr}(L)}, \quad \overline{\exp}^{gr}(L) = \limsup_{n \rightarrow \infty} \sqrt[n]{c_n^{gr}(L)}$$

called the *lower* and *upper graded PI-exponents* of  $L$ , respectively. If an ordinary limit exists, it is called an *(ordinary) graded PI-exponent* of  $L$ ,

$$\exp^{gr}(L) = \lim_{n \rightarrow \infty} \sqrt[n]{c_n^{gr}(L)}.$$

Symmetric groups and their representations play an important role in the theory of codimensions. One can find all details concerning application of representation theory of symmetric groups to study of polynomial identities in [1], [3], [7]. In case of graded identities one can consider  $(S_k \times S_{n-k})$ -action on multilinear graded polynomials. Namely, the subspace  $P_{k, n-k} \subseteq \mathcal{L}(X, Y)$  has a natural structure of  $(S_k \times S_{n-k})$ -module where  $S_k$  acts on even variables  $x_1, \dots, x_k$  while  $S_{n-k}$  acts on

odd variables  $y_1, \dots, y_{n-k}$ . Clearly,  $P_{k,n-k} \cap Id^{gr}(L)$  is the submodule under this action and we get an induced  $S_k \times S_{n-k}$ -action on  $P_{k,n-k}(L)$ . If  $G$  is a subgroup of  $S_k \times S_{n-k}$  then  $G$  also acts naturally on  $P_{k,n-k}(L)$ . In particular,

$$(1) \quad c_{k,n-k}(L) \geq \dim M$$

for any subgroup  $G \subseteq S_k \times S_{n-k}$  and for any  $G$ -submodule  $M$  of  $P_{k,n-k}(L)$ . We will use the relation (1) for getting a lower bound of  $c_{k,n-k}(L)$ .

### 3. GRADED PI-EXPONENT OF $b(2)$

Recall the construction of Lie superalgebra  $L = L_0 \oplus L_1 = b(2)$ . Even component  $L_0$  consists of all  $4 \times 4$  matrices of the type

$$L_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix} \mid A \in M_2(F), \text{tr}(A) = 0 \right\},$$

where  $A$  is  $2 \times 2$  traceless matrix and  $t : A \rightarrow A^t$  is the usual transpose involution.

Odd component  $L_1$  consists of  $4 \times 4$  matrices

$$L_1 = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \mid \text{where } B^t = B, C^t = -C \quad B, C \in M_2(F) \right\}.$$

Also denote

$$L_1^- = \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \mid C^t = -C \in M_2(F) \right\},$$

$$L_1^+ = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \mid B^t = B \in M_2(F) \right\}.$$

Then  $\dim L_1^- = 1, \dim L_1^+ = 3$ .

As a vector space  $L$  is embedded into  $M_4(F)$ . Using ordinary associative matrix multiplication we can define super-Lie product on  $L$  as

$$\{x, y\} = xy - (-1)^{|x||y|}yx$$

for any homogeneous  $x, y \in L_0 \cup L_1$  where  $|x| = 0$  if  $x \in L_0$  while  $|x| = 1$ , if  $x \in L_1$ .

Note that  $L_0$  is a Lie algebra isomorphic to  $sl_2(F)$ . We will identify  $L_0$  with  $sl_2(F)$  and use the standard basis of  $sl_2(F)$

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Furthermore we will not use associative multiplication. This will allow us to omit super-Lie brackets, i.e. to write  $ab$  instead of  $\{a, b\}$ . We will also use the notation  $ab \cdots c$  for the left-normed product  $\{\{\dots \{a, b\}, \dots\}, c\}$ .

We will also use the following agreement for denoting alternating sets of variables. If  $f = f(x_1, \dots, x_k, y_1, \dots, y_n)$  is a multilinear polynomial and we apply to  $f$  the operator of alternation on variables  $x_1, \dots, x_k$ , then we will write the same symbol (bar, double bar, tilde, double tilde, etc.) over the variables  $x_1, \dots, x_k$ , that is

$$f(\bar{x}_1, \dots, \bar{x}_k, y_1, \dots, y_n) = \sum_{\sigma \in S_k} (\text{sgn } \sigma) f(x_{\sigma(1)}, \dots, x_{\sigma(k)}, y_1, \dots, y_n).$$

For example  $\bar{x}a\bar{y} = xay - yax$ , or

$$\bar{\bar{x}}_1 a \bar{\bar{x}}_2 b \bar{\bar{x}}_3 = \sum_{\sigma \in S_3} (\text{sgn } \sigma) x_{\sigma(1)} a x_{\sigma(2)} b x_{\sigma(3)}.$$

We will also use this notation for non-multilinear polynomials with repeating variables as follows

$$\begin{aligned}\bar{x}_1\bar{x}_2a\bar{x}_1\bar{x}_2 &= x_1x_2a\bar{x}_1\bar{x}_2 - x_2x_1a\bar{x}_1\bar{x}_2 = \\ &= x_1x_2ax_1x_2 - x_1x_2ax_2x_1 - x_2x_1ax_1x_2 + x_2x_1ax_2x_1.\end{aligned}$$

Following this agreement we consider an alternating expression  $h\bar{e}\bar{f}\bar{h}$  in the Lie algebra  $L_0 = sl_2(F)$ . Since  $he = 2e, hf = -2f, ef = h$ , it easily follows that

$$h\bar{e}\bar{f}\bar{h} = 8h$$

and

$$(2) \quad h \underbrace{\bar{e}\bar{f}\bar{h}\bar{e}\bar{f}\bar{h} \cdots \bar{e}\bar{f}\bar{h}}_{t \text{ alternating triples}} = 8^t h.$$

Consider a multilinear polynomial

$$g = g(x_0, x_1^1, x_2^1, x_3^1, \dots, x_1^t, x_2^t, x_3^t) = Alt_1 Alt_2 \dots Alt_t(x_0 x_1^1 \cdots x_3^1)$$

where  $Alt_j$  is the operator of alternation on  $x_1^j, x_2^j, x_3^j$ ,  $1 \leq j \leq t$ , that is

$$g = x_0 \bar{x}_1^1 \bar{x}_2^1 \bar{x}_3^1 \cdots \bar{x}_1^t \bar{x}_2^t \bar{x}_3^t.$$

The evaluation  $\varphi : x_1^1, \dots, x_1^t \rightarrow e, x_2^1, \dots, x_2^t \rightarrow f, x_0, x_3^1, \dots, x_3^t \rightarrow h$  gives us

$$(3) \quad \varphi(g) = 8^t h$$

in  $L_0$  by (2). Moreover, if we denote the symmetrization on variables  $x_i^1, \dots, x_i^t$ ,  $i = 1, 2, 3$  by  $Sym_i$ , then it follows from (3) and the definition of  $\varphi$  that

$$(4) \quad \varphi(Sym_1 Sym_2 Sym_3(g)) = (t!)^3 8^t h$$

in  $L_0$  by virtue of (2). An element

$$(5) \quad g' = Sym_1 Sym_2 Sym_3(g)$$

with the fixed  $x_0$  generates in  $P_{3t+1,0}$  an irreducible  $S_{3t}$ -submodule with the character  $\chi_\lambda, \lambda = (t, t, t)$  where the permutation group  $S_{3t}$  acts on  $x_1^1, x_2^1, x_3^1, \dots, x_1^t, x_2^t, x_3^t$ .

Given a partition  $\mu = (\mu_1, \dots, \mu_d) \vdash n$ , we define the function

$$\Phi(\mu) = \frac{1}{z_1^{z_1} \cdots z_d^{z_d}},$$

where

$$z_1 = \frac{\mu_1}{n}, \dots, z_d = \frac{\mu_d}{n}.$$

The value of  $\Phi(\mu)$  is closely connected with  $\deg \chi(\mu)$ .

**Lemma 1.** [8, Lemma 1] *Let  $n \geq 100$ . Then*

$$\frac{\Phi(\mu)^n}{n^{d^2+d}} \leq d_\mu \leq n\Phi(\mu)^n.$$

□

In particular, if  $m = 3t$  and  $\mu = (t, t, t)$  then

$$(6) \quad \deg \chi_\mu \geq m^{-12} 3^m.$$

In the next step we will construct an irreducible  $S_{6k}$ -submodule in  $P_{1,6k+1} \not\subset Id^{gr}(L)$  where  $S_{6k}$  acts on some  $6k$  odd variables. Denote

$$d = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \in L_1^-.$$

Then  $L_1^- = Span \langle d \rangle$ . It is not difficult to check that  $L_1^+ L_1^- = L_0$ . Hence there exist  $a, b, c \in L_1^+$  such that

$$ad = e, \quad bd = f, \quad cd = h$$

where  $\{e, f, h\}$  is the fixed basis of  $L_0$ . It follows that

$$h(\bar{a}d)(\bar{b}d)(\bar{c}d) = 8h$$

in  $L$  and

$$h(\bar{a}d)(\bar{b}d)(\bar{c}d)(\bar{\bar{a}}\bar{d})(\bar{\bar{b}}\bar{d})(\bar{\bar{c}}\bar{d}) = h(\bar{a}d)(\bar{b}d)(\bar{c}d)(\bar{\bar{a}}\bar{d})(\bar{\bar{b}}\bar{d})(\bar{\bar{c}}\bar{d}) = 64h.$$

Repeating this procedure and using (3) we obtain a multialternating expression (7)

$$H = h(\bar{a}d)(\bar{b}d)(\bar{c}d)(\bar{\bar{a}}\bar{d})(\bar{\bar{b}}\bar{d})(\bar{\bar{c}}\bar{d})(\tilde{a}\tilde{d})(\tilde{b}\tilde{d})(\tilde{c}\tilde{d}) \cdots (\tilde{\tilde{a}}\tilde{\tilde{d}})(\tilde{\tilde{b}}\tilde{\tilde{d}})(\tilde{\tilde{c}}\tilde{\tilde{d}})(ad)(bd)(cd) = 8^{k+1}h$$

depending on one  $h$ ,  $k+1$  elements  $a, b, c$  and  $3(k+1)$  elements  $d$ . The element  $H$  on the left hand side of (7) contains  $k$  alternating sets  $\{a, b, c, d\}$ . The first set consists of 1st  $a$ , 1st  $b$ , 1st  $c$  and 4th  $d$ . The second set consists of 2nd  $a$ , 2nd  $b$ , 2nd  $c$  and 7th  $d$ , and so on. The element  $H$  also contains  $2(k+1)+1$  non-alternating entries  $d$  and four extra factors  $h, a, b, c$  out of alternating sets. This  $H$  is a value of the following multilinear polynomial: denote by

$$w = w(x_0, y_1^1, y_2^1, y_3^1, z_1^1, z_2^1, z_3^1, \dots, y_1^{k+1}, y_2^{k+1}, y_3^{k+1}, z_1^{k+1}, z_2^{k+1}, z_3^{k+1}) = \\ Alt_1 \dots Alt_k (x_0(y_1^1 z_1^1)(y_2^1 z_2^1)(y_3^1 z_3^1)(y_1^2 z_1^2)(y_2^2 z_2^2)(y_3^2 z_3^2) \cdots \\ (y_1^{k+1} z_1^{k+1})(y_2^{k+1} z_2^{k+1})(y_3^{k+1} z_3^{k+1}))$$

where  $x_0$  is an even variable, all  $y_\beta^\alpha, z_\beta^\alpha$  are odd and  $Alt_j$  is the operator of alternation on  $y_1^j, y_2^j, y_3^j, z_1^{j+1}, j = 1, \dots, k$ . Then  $\varphi(w) = H$  where  $\varphi$  is an evaluation of the form

$$\varphi(x_0) = h, \varphi(y_1^j) = a, \varphi(y_2^j) = b, \varphi(y_3^j) = c, \varphi(z_i^j) = d, j = 1, \dots, k+1, i = 1, 2, 3.$$

Also denote by  $Sym_1, Sym_2, Sym_3$  the symmetrization on the sets  $\{y_1^1, \dots, y_1^k\}, \{y_2^1, \dots, y_2^k\}, \{y_3^1, \dots, y_3^k\}$ , respectively, and by  $Sym_4$  the symmetrization on  $\{z_1^2, z_2^2, z_3^2, \dots, z_1^{k+1}, z_2^{k+1}, z_3^{k+1}\}$ . If

$$w' = Sym_1 Sym_2 Sym_3 Sym_4(w)$$

then

$$\varphi(w') = (3k)!(k!)^3 \varphi(w) = (3k)!(k!)^3 8^{k+1}h$$

in  $L$ . In particular,  $w'$  is not an identity of  $L$ .

Now let the permutation group  $S_{6k}$  act on the set  $\{y_i^j, z_i^{j+1} | 1 \leq j \leq k, i = 1, 2, 3\}$  and let  $x_0, z_1^1, z_2^1, z_3^1, y_1^{k+1}, y_2^{k+1}, y_3^{k+1}$  be fixed. Then  $w'$  generates an irreducible  $S_{6k}$ -submodule in  $P_{1,6(k+1)}$  corresponding to the partition  $\mu = (3k, k, k, k) \vdash r = 6k$  with

$$\Phi(\mu) = 2\sqrt{3}.$$

Hence

$$(8) \quad \deg \chi_\mu \geq r^{-20}(2\sqrt{3})^r$$

by Lemma 1.

Now let

$$u = w'(g', y_1^1, y_2^1, y_3^1, \dots, z_1^{k+1}, z_2^{k+1}, z_3^{k+1})$$

where  $g'$  is taken from (5). If  $m = 3t, r = 6k$ , group  $S_m$  acts on even variables  $x_i^j$  from  $g'$  whereas  $S_r$  acts on odd variables  $\{y_i^j, z_i^j\}$  (except  $z_1^1, z_2^1, z_3^1, y_1^{k+1}, y_2^{k+1}, y_3^{k+1}$ ) then  $u$  is not a graded identity of  $L$  as follows from (4), (6) and (7) and it generates an irreducible  $S_m \times S_r$ -submodule  $M$  in  $P_{m+1, r+6}$  with the character  $\chi_{\lambda, \mu}$  where  $\lambda = (t, t, t), \mu = (3k, k, k, k)$ . Hence by (6), (8)

$$\dim M = \deg \chi_\lambda \deg \chi_\mu \geq \frac{1}{(m+r)^{32}} 3^m (2\sqrt{3})^r$$

and then by (1)

$$(9) \quad c_{m+1, r+6}(L) = \dim P_{m+1, r+6}(L) \geq \dim M \geq \frac{3^m (2\sqrt{3})^r}{(m+r)^{32}}.$$

The inequality (9) means that we have proved the following lemma.

**Lemma 2.** *Let  $t, r \geq 1$  be arbitrary integers and  $m = 3t, r = 6k$ . Then*

$$c_{m+1, r+6}(L) \geq \frac{3^m (2\sqrt{3})^r}{(m+r)^{32}}.$$

□

Now we will find a lower bound for  $n$ th graded codimension of  $L$  for the special case of  $n$ .

**Lemma 3.** *Let  $n - 7$  be a multiple of 6. Then*

$$c_n^{gr}(L) \geq \frac{1}{3^{18n} 38} (3 + 2\sqrt{3})^n.$$

*Proof.* Let  $q = n - 7$ . Then applying Lemma 2 we obtain

$$\begin{aligned} c_n^{gr}(L) &= \sum_i \binom{n}{i} c_{i, n-i}(L) \geq \sum_{j=0}^{q/6} \binom{n}{6j+1} c_{1+6j, q+6-6j}(L) \geq \\ &\frac{1}{q^{32}} \sum_{j=0}^{q/6} \binom{n}{6j+1} 3^{6j} (2\sqrt{3})^{q-6j} \geq \frac{A}{n^{33}} \end{aligned}$$

where

$$A = \sum_{j=0}^{q/6} \binom{n}{6j} 3^{6j} (2\sqrt{3})^{q-6j}$$

since

$$\binom{n}{i} \leq n \binom{n}{i+1}.$$

Now, since given  $0 \leq j < \frac{q}{6}$ , we have

$$\binom{q}{6j+i} 3^{6j+i} (2\sqrt{3})^{q-6j-i} < (3q)^5 \binom{q}{6j} 3^{6j} (2\sqrt{3})^{q-6j}$$

for all  $1 \leq i \leq 5$ . It follows that

$$A > \frac{6}{(3n)^5} \sum_{i=0}^q \binom{q}{i} 3^i (2\sqrt{3})^{q-i} = \frac{6}{(3n)^5} (3 + 2\sqrt{3})^q > \frac{2}{3^{18}n^5} (3 + 2\sqrt{3})^n.$$

Hence

$$c_n^{gr}(L) > \frac{(3 + 2\sqrt{3})^n}{3^{18}n^{38}}.$$

□

Now we consider the case when  $n-7$  is not a multiple of 6, that is  $n-7 \equiv i \pmod{6}$  with  $1 \leq i \leq 5$ .

**Lemma 4.** *Let  $t, r, i \geq 1$  be arbitrary integers,  $m = 3t$ ,  $r = 6k$  and  $i \leq 5$ . Then*

$$c_{m+1+i, r+6}(L) \geq \frac{3^m (2\sqrt{3})^r}{(m+r)^{32}}.$$

*Proof.* The proof is similar to the proof of Lemma 2. We only need to change the polynomial  $u = w(g', y_1^1, \dots, z_3^{k+1})$  to  $u' = ux_1 \cdots x_i$  and consider an evaluation  $\varphi$  with the same values on  $x_0, x_j^i, y_j^i, z_j^i$  as in Lemma 2 and  $\varphi(x_1) = e$ ,  $\varphi(x_2) = \dots = \varphi(x_i) = h$  if  $i \geq 2$ . Then  $\varphi(u') = \pm 2^i \varphi(u) \neq 0$  and we are done. □

Slightly modifying arguments of Lemma 3 and using Lemma 4, we get the following result for arbitrary  $1 \leq i \leq 5$ .

**Lemma 5.** *Let  $n-7 \equiv i \pmod{6}$ ,  $1 \leq i \leq 5$ . Then*

$$c_n^{gr}(L) \geq \frac{1}{3^{18}n^{43}} (3 + 2\sqrt{3})^n.$$

□

Now we are ready to prove the main result of the paper.

**Theorem 1.** *Graded PI-exponent of Lie superalgebra  $L = b(2)$  exists and is equal to*

$$\exp^{gr}(L) = 3 + 2\sqrt{3}.$$

*Proof.* By Lemmas 3 and 5

$$\underline{\exp}^{gr}(L) = \liminf_{n \rightarrow \infty} \sqrt[n]{c_n^{gr}(L)} \geq 3 + 2\sqrt{3}.$$

On the other hand,  $\overline{\exp}^{gr}(b(t)) \leq t^2 - 1 + t\sqrt{t^2 - 1}$  for all  $t \geq 2$  as proved in [10]. Hence the limit

$$\exp^{gr}(b(2)) = \lim_{n \rightarrow \infty} \sqrt[n]{c_n^{gr}(b(2))}$$

exists and

$$\exp^{gr}(L) = 3 + 2\sqrt{3}.$$

□

## REFERENCES

- [1] Yu. A. Bahturin, Identical Relations in Lie Algebras, Utrecht, VNU Science Press, 1987.
- [2] Yu. A. Bahturin, V. Drensky, Graded polynomial identities of matrices, Linear Algebra Appl. 357 (2002), 15-34.
- [3] V. Drensky, Free Algebras and PI-Algebras, Springer-Verlag, Berlin, 2000.
- [4] A. Giambruno, I. Shestakov, M. Zaicev, Finite-dimensional non-associative algebras and codimension growth, Adv. Appl. Math. 47 (2011), 125-139.

- [5] A. Giambruno, M. Zaicev, On codimension growth of finitely generated associative algebras, *Adv. Math.* 140 (1998), 145-155.
- [6] A. Giambruno, M. Zaicev, Exponential codimension growth of PI algebras: an exact estimate, *Adv. Math.* 142 (1999), 221-243.
- [7] A. Giambruno, M. Zaicev, *Polynomial Identities and Asymptotic Methods*. Math. Surveys Monogr., vol. 12, Amer. Math. Soc., Providence, RI, 2005.
- [8] A. Giambruno, M. Zaicev, On codimension growth of finite-dimensional Lie superalgebras, *J. Lond. Math. Soc.* (2) 85 (2012), 534-548.
- [9] M. Scheunert, *The theory of Lie superalgebras; an introduction*, Lecture Notes in Math., vol. 716, Springer-Verlag, Berlin, 1979.
- [10] D. Repovš, M. Zaicev, Graded identities of some simple Lie superalgebras.- *Algebr. Represent. Theor.* 17 (2014), 1401-1412.
- [11] M. Zaicev, Integrality of exponents of growth of identities of finite-dimensional Lie algebras, (Russian), *Izv. Ross. Akad. Nauk Ser. Mat.* 66 (2002), 23-48; English translation: *Izv. Math.* 66 (2002), 463-487.
- [12] M.V. Zaytsev, On existence of PI-exponents of growth of identities. International Conference "Mal'tsev Meeting. Novosibirsk, November 11-15, 2013", Collection of Abstracts, p. 120.

DUŠAN REPOVŠ, FACULTY OF EDUCATION, AND FACULTY OF MATHEMATICS AND PHYSICS, UNIVERSITY OF LJUBLJANA, P. O. B. 2964, LJUBLJANA, 1001, SLOVENIA

*E-mail address:* `dusan.repovs@guest.arnes.si`

MIKHAIL ZAICEV, DEPARTMENT OF ALGEBRA, FACULTY OF MATHEMATICS AND MECHANICS, MOSCOW STATE UNIVERSITY, MOSCOW, 119992, RUSSIA

*E-mail address:* `zaicevmv@mail.ru`